Midterm Solution

1. Evaluate $\int_{C_e(0)} \frac{ze^z}{z-i}dz$ ***and*** $\int_{C_e(0)} \frac{ze^z}{z-e^2 i}dz$.

Answer: Let $f(z) = ze^z$. ***Clearly,*** f ***is holomorphic on*** $\mathbb{C}$. ***Let*** U ***be an open region containing*** $C_e(0)$ ***and its interior and*** $U \subset C_{e^2}(0)$.
(i) ***By Cauchy's integral formula, we have*** $\int_{C_e(0)} \frac{ze^z}{z-i}dz = f(i) = ie^i$.
(ii) ***Since*** $\frac{ze^z}{z-e^2 i}$ ***is holomorphic on*** U, ***therefore*** $\int_{C_{e^2}(0)} \frac{ze^z}{z-e^2 i}dz = 0$.

2. Let $\overline{B_1(0)} \subset U$ ***and*** $f(z) = \sum_{n=0}^{\infty} a_n z^n \in Hol(U)$ ***and that*** $|f(z)| \leq 1$ ***for all*** $z \in C_1(0)$. ***Prove that*** $|a_n| \leq 1$ ***for all*** n.

Answer: Here we use Cauchy's inequalities, i.e.,

$$|\frac{f^{(n)}(0)}{n!}| \leq \|f\|,$$

where $\|f\| = \sup_{z \in C_1(0)} |f(z)|$. ***It is given that*** $\|f\| \leq 1$. ***Again*** $a_n = \frac{f^{(n)}(0)}{n!}$ ***for all*** n. ***Therefore*** $|a_n| \leq 1$ ***for all*** n.

3. Let $\overline{B_r(0)} \subset U$ ***and*** $f \in Hol(U)$. ***then prove that*** $\int_{C_r(0)} \frac{f'(\zeta)}{\zeta - z}d\zeta = \int_{C_r(0)} \frac{f(\zeta)}{(\zeta - z)^2}d\zeta$ ***for*** $z \in B_r(0)$.

Answer: Since $f \in Hol(U)$, $f' \in Hol(U)$. ***Here we use Cauchy's integral formula: For*** $n \geq 0$

$$\frac{f^{(n)}(0)}{n!} = \frac{1}{2\pi i} \int_{C_r(0)} \frac{f'(\zeta)}{(\zeta - z)^{n+1}} d\zeta$$

for all z ***in the interior of*** $C_r(0)$.
We have $\int_{C_r(0)} \frac{f'(\zeta)}{\zeta - z}d\zeta = \frac{2\pi i}{n!} f'(z)$ ***and*** $\int_{C_r(0)} \frac{f(\zeta)}{(\zeta - z)^2}d\zeta = \frac{2\pi i}{n!} f'(z)$.
This proves that the two integrals are same.

4. Expand $\frac{1}{(1-z)^2}$ ***in a series of powers of*** z ***and find the radius of convergence.***

Answer. Here $\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} nz^n$. ***Therefore the radius of convergence of the power series is*** 1 ***as*** $\lim_{n \to \infty} n^{\frac{1}{n}} = 1$.

5. Let $f \in Hol(\mathbb{C})$. ***What can you conclude about*** f***: (i) when*** $f(\mathbb{C} \cap B_1(0)$ ***is an empty set. (ii) when*** f, ***restricted to*** $\mathbb{R}$, ***is a*** 2π***-periodic function.***

Answer. Let $f \in Hol(\mathbb{C})$. ***If*** f ***is non constant entire function then by Picard's little theorem we have either*** $\mathbb{C} \backslash f(\mathbb{C})$ ***is empty or singleton. But*** $f(\mathbb{C}) \cap B_1(0)$ ***is empty so*** f ***must be constant on*** $\mathbb{C}$.

Let $g(z) = f(z + 2\pi)$ ***for*** $z \in \mathbb{C}$. ***Since*** $f \in Hol(\mathbb{C})$, $g \in Hol(\mathbb{C})$. ***It is given that*** f, ***restricted to*** $\mathbb{R}$, ***is a*** 2π***-periodic function i.e.,*** $f(x + 2\pi) = f(x)$ ***for*** $x \in \mathbb{R}$. ***Consider the set*** $\mathcal{D} = \{(x, 0) : g(x) = f(x + 2\pi) = f(x)\}$. ***Then*** $\mathcal{D} \subset \mathbb{C}$ ***and every point of*** $\mathcal{D}$ ***is a limit point of*** $\mathbb{C}$. ***By Identity theorem we have*** $g(z) = f(z)$ ***i.e.*** $f(z + 2\pi) = f(z)$ ***for all*** $z \in \mathbb{C}$.

6. Let $U \subset \mathbb{C}$ be a domain. Let $f, g \in Hol(U)$ and that $fg \equiv 0$. Prove that either $f \equiv 0$ or $g \equiv 0$.

Answer. Let $z_0 \in U$. Then $f(z_0)g(z_0) = 0$. This implies either $f(z_0) = 0$ or $g(z_0) = 0$. Suppose $g(z_0) \neq 0$. Then there exists $r > 0$ such that $g(z) \neq 0$ for all $z \in B_r(z_0)$. But $f(z)g(z) = 0$ for all $z \in B_r(z_0)$. Therefore $f(z) = 0$ for all $z \in B_r(z_0)$. Hence z_0 is a limit point of $\{z \in U : f(z) = 0\}$. Now using Identity theorem, we have $f \equiv 0$. This completes the proof.

7. Suppose f and g are continuous functions on $\overline{B_r(0)}$ and $f, g \in Hol(B_r(0))$ and that $f(z) \neq 0$ and $g(z) \neq 0$ for all $z \in \overline{B_r(0)}$. If $|f(z)| = |g(z)|$ for all $z \in C_r(0)$, then show that there exists a constant c such that $|c| = 1$ and $f(z) = cg(z)$ for all $z \in B_r(0)$.

Answer: Consider $h(z) = \frac{f(z)}{g(z)}$ for all $z \in \overline{B_r(0)}$. Then h is continuous on $\overline{B_r(0)}$ and holomorphic on $B_r(0)$. By Maximum Modulus principle, h attains maximum on the boundary $C_r(0)$. Here $|h(z)| = 1$ on $C_r(0)$. Similarly $\frac{1}{h}$ is defined on $\overline{B_r(0)}$. Applying Maximum Modulus principle on $\frac{1}{h}$, we have $|\frac{1}{h(z)}| = 1$ on $C_r(0)$. h attains both maximum and minimum on the boundary and hence h must be constant c with $|c| = 1$. Therefore $f(z) = cg(z)$ for all $z \in B_r(0)$.

8. Let $\Omega \subset R^2$ be an open set and $u, v \in C^1(\Omega)$. Assume that u and v satisfy the Cauchy-Riemann equations in Ω. Assume moreover that $u(x,y)^2 + v(x,y)^2 \neq 0$ for all $(x,y) \in \Omega$. Show that the function

$$\frac{uu_x + vv_x}{u^2 + v^2}$$

is harmonic in Ω.

Answer. Since u and v satisfy the Cauchy-Riemann equations i.e. $u_x = v_y$ and $u_y = -v_x$ in Ω, we have $u_{xx} + u_{yy} = 0$ and $v_{xx} + v_{yy} = 0$. Therefore u, v are harmonic in Ω. Consider $f = u^2 + v^2$ and $h = \frac{uu_x + vv_x}{u^2 + v^2}$.

$$h = \frac{1}{2}\frac{f_x}{f}.$$

Now by easy calculation, we have $h_{xx} + h_{yy} = 0$. Thus h is harmonic in Ω.

9.Prove that

$$f(z) := \int_0^1 \frac{sinzt}{t} dt \qquad (z \in \mathbb{C}),$$

is an entire function.

Answer: The series expansion of

$$\begin{aligned} \frac{sinzt}{t} &= \frac{1}{t}[zt - \frac{(zt)^3}{3!} + \frac{(zt)^5}{5!} - \ldots] \\ &= [z - \frac{z^3 t^2}{3!} + \frac{z^5 t^4}{5!} - \ldots]. \end{aligned}$$

Since $\frac{sinzt}{t}$ is again a power series with radius of convergence ∞, term by term integration we have

$$\begin{aligned} f(z) = \int_0^1 \frac{sinzt}{t} dt &= [z - \frac{z^3}{3!3} + \frac{z^5}{5!5} - \ldots] \\ &= \sum_{n=0}^{\infty} a_n z^n, \end{aligned}$$

where $a_{2n+1} = \frac{1}{(2n+1)!(2n+1)}$ ***and*** $a_{2n} = 0$ ***for*** $n \geq 0$***.***

Clearly $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = 0$***. Therefore*** $f(z)$ ***is an entire function on*** $\mathbb{C}$***.***